

Chapter 2

Inverse Trigonometric Functions

Basics Concepts: Inverse Trigonometric Functions

What is an Inverse Function?

- A function accepts values, performs particular operations on these values, and generates an output. The inverse function agrees with the resultant, operates, and reaches back to the original function.
- If $y=f(x)$ and $x=g(y)$ are two functions such that $f(g(y)) = y$ and $g(f(y)) = x$, then f and y are said to be inverse of each other.
Example: If $f(x) = 2x + 5 = y$, then, $g(y) = (y-5)/2 = x$ is the inverse of $f(x)$.
- The inverse of f is denoted by f^{-1}

Inverse Trigonometric Functions

- Trigonometric functions are **many-one functions** but we know that inverse of function exists if the function is bijective.
- If we restrict the domain of trigonometric functions, then these functions become bijective and the inverse of trigonometric functions are defined within the restricted domain.
- **Example:** $y = f(x) = \sin x$, then its inverse is $x = \sin^{-1} y$.

Inverse Trigonometric Formulas



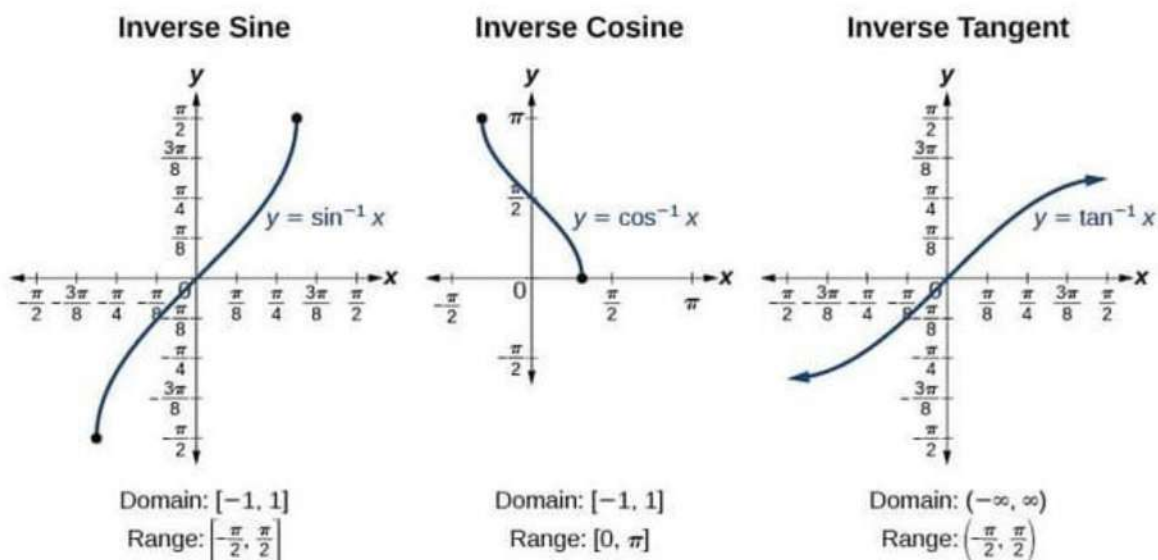


Fig: Inverse Trigonometric Functions

The inverse trigonometric functions are the inverse functions of the trigonometric functions written as $\cos^{-1} x$, $\sin^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\operatorname{cosec}^{-1} x$, $\sec^{-1} x$.

The inverse trigonometric functions are multivalued. For example, there are multiple values of ω such that $z = \sin \omega$, so $\sin^{-1} z$ is not uniquely defined unless a principal value is defined.

Such principal values are sometimes denoted with a capital letter so, for example, the principal value of the inverse sine may be variously denoted $\sin^{-1} z$ or $\operatorname{arcsin} z$.

Let's say, if $y = \sin x$, then $x = \sin^{-1} y$, similarly for other trigonometric functions. This is one of the inverse trigonometric formulas. Now, $y = \sin^{-1}(x)$, $y \in [-\pi/2, \pi/2]$ and $x \in [-1, 1]$.

- Thus, $\sin^{-1} x$ has infinitely many values for given $x \in [-1, 1]$.
- There is only one value among these values which lies in the interval $[-\pi/2, \pi/2]$. This value is called the principal value.

Domain and Range of Inverse Trigonometric Formulas

Function	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$

$\tan^{-1}x$	R	$(-\pi/2, \pi/2)$
$\cot^{-1}x$	R	$(0, \pi)$
$\sec^{-1}x$	$R - (-1, 1)$	$[0, \pi] - \{\pi/2\}$
$\operatorname{cosec}^{-1}x$	$R - (-1, 1)$	$[-\pi/2, \pi/2] - \{0\}$

Solved Examples

Ques 1: Find the exact value of each expression without a calculator, in $[0, 2\pi)$.

- $\sin^{-1}(-3\sqrt{2})$
- $\cos^{-1}(-2\sqrt{2})$
- $\tan^{-1}\sqrt{3}$

Ans:

- Recall that $-3\sqrt{2}$ is from the 30–60–90 triangle. The reference angle for $\sin$ and $3\sqrt{2}$ would be 60° . Because this is sine and it is negative, it must be in the third or fourth quadrant. The answer is either $4\pi/3$ or $5\pi/3$.
- $-2\sqrt{2}$ is from an isosceles right triangle. The reference angle is then 45° . Because this is cosine and negative, the angle must be in either the second or third quadrant. The answer is either $3\pi/4$ or $5\pi/4$.
- $\sqrt{3}$ is also from a 30–60–90 triangle. Tangent is $\sqrt{3}$ for the reference angle 60° . Tangent is positive in the first and third quadrants, so the answer would be $\pi/3$ or $4\pi/3$.

Notice how each one of these examples yields two answers. This poses a problem when finding a singular inverse for each of the trig functions. Therefore, we need to restrict the domain in which the inverses can be found.

Ques 2: Find the value of $\tan^{-1}(1.1106)$.

Ans: Let $A = \tan^{-1}(1.1106)$

Then, $\tan A = 1.1106$

$A = 48^\circ$

$\tan 48 = 1.1106$

[Use calculator in degree mode]

$$\tan^{-1} 1.1106 = 48^\circ$$

Properties of Inverse

Here are the properties of the inverse trigonometric functions with proof.

Property 1

1. $\sin^{-1}(1/x) = \operatorname{cosec}^{-1}x$, $x \geq 1$ or $x \leq -1$
2. $\cos^{-1}(1/x) = \sec^{-1}x$, $x \geq 1$ or $x \leq -1$
3. $\tan^{-1}(1/x) = \cot^{-1}x$, $x > 0$

Proof: $\sin^{-1}(1/x) = \operatorname{cosec}^{-1}x$, $x \geq 1$ or $x \leq -1$,

Let $\sin^{-1}x = y$

i.e. $x = \operatorname{cosec} y$

$$1/x = \sin y$$

$$\sin^{-1}\left(\frac{1}{x}\right) = y$$

$$\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x$$

$$\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x$$

Hence, $\sin^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}x$ where, $x \geq 1$ or $x \leq -1$.

Property 2

1. $\sin^{-1}(-x) = -\sin^{-1}(x)$, $x \in [-1, 1]$
2. $\tan^{-1}(-x) = -\tan^{-1}(x)$, $x \in \mathbb{R}$
3. $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x)$, $|x| \geq 1$

Proof: $\sin^{-1}(-x) = -\sin^{-1}(x)$, $x \in [-1, 1]$

Let, $\sin^{-1}(-x) = y$

Then $-x = \sin y$

$$x = -\sin y$$

$$x = \sin(-y)$$

$$\sin^{-1} = \sin^{-1}(\sin(-y))$$

$$\sin^{-1}x = y$$

$$\sin^{-1}x = -\sin^{-1}(-x)$$

$$\text{Hence, } \sin^{-1}(-x) = -\sin^{-1}x \in [-1, 1]$$

Property 3

1. $\cos^{-1}(-x) = \pi - \cos^{-1}x, x \in [-1, 1]$
2. $\sec^{-1}(-x) = \pi - \sec^{-1}x, |x| \geq 1$
3. $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in \mathbb{R}$

$$\text{Proof : } \cos^{-1}(-x) = \pi - \cos^{-1}x, x \in [-1, 1]$$

$$\text{Let } \cos^{-1}(-x) = y$$

$$\cos y = -x \quad x = -\cos y$$

$$x = \cos(\pi - y)$$

$$\text{Since, } \cos \pi - q = -\cos q$$

$$\cos^{-1}x = \pi - y$$

$$\cos^{-1}x = \pi - \cos^{-1}(-x)$$

$$\text{Hence, } \cos^{-1}(-x) = \pi - \cos^{-1}x$$

Solved Example

Ques 1: Prove that " $\sin^{-1}(-x) = -\sin^{-1}(x), x \in [-1, 1]$ "

Ans: Let, $\sin^{-1}(-x) = y$

$$\text{Then } -x = \sin y$$

$$x = -\sin y$$

$$x = \sin(-y)$$

$$\sin^{-1}x = \arcsin(\sin(-y))$$

$$\sin^{-1} x = y$$

$$\sin^{-1} x = -\sin^{-1}(-x)$$

$$\text{Hence, } \sin^{-1}(-x) = -\sin^{-1} x, x \in [-1, 1]$$

This concludes our discussion on the topic of trigonometric inverse functions.

Ques 2: $\sin^{-1}(\cos \pi/3) = ?$

$$\text{Ans: } \sin^{-1} \left(\cos \frac{\pi}{3} \right) = \sin^{-1} \frac{1}{2} \text{ [substitute } \cos(\pi/3) = 1/2]$$

$$= \pi/6 \text{ [substitute } \sin^{-1}(1/2) = \pi/6]$$

Ques 3: Find the value of $\sin(\pi/4 + \cos^{-1}(\sqrt{2}/2))$.

Ans:

$$\text{Let } y = \sin \left(\frac{\pi}{4} + \cos^{-1} \left(\frac{\sqrt{2}}{2} \right) \right) \text{ and } A = \cos^{-1} \left(\frac{\sqrt{2}}{2} \right)$$

$$\text{Then, } \cos A = \sqrt{2}/2$$

Multiplying the numerator as well as denominator by $\sqrt{2}$ we get:

$$\cos A = 1/\sqrt{2}$$

$$A = \pi/4$$

Therefore

$$y = \sin \left(\frac{\pi}{4} + \frac{\pi}{4} \right)$$

$$y = \sin(\pi/2)$$

$$\text{hence, } y = 1.$$

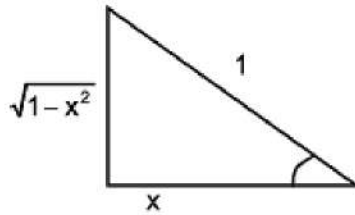
Solved Examples: Inverse Trigonometric Functions

Equations Involving Inverse Trigonometric Functions

Ex.1 Solve $\cos^{-1} x \sqrt{3} + \cos^{-1} x = \pi/2$.

Sol.





Given $\cos^{-1} x\sqrt{3} + \cos^{-1} x = \pi/2$ (1)

or $\cos^{-1} x\sqrt{3} = \pi/2 - \cos^{-1} x$

or $\cos \cos^{-1} x\sqrt{3} = \cos (\pi/2 - \cos^{-1} x)$

or $x\sqrt{3} = \sin \cos^{-1} x$ or

$x\sqrt{3} = \sin \sin^{-1} \sqrt{1-x^2}$

or $x\sqrt{3} = \sqrt{1-x^2}$

Squaring we get $3x^2 = 1 - x^2$ or $4x^2 = 1 = x = \pm 1/2$

Verification : When $x = 1/2$

L.H.S. of equation = $\cos^{-1} (3/2) + \cos^{-1} (1/2) = \pi/6 + \pi/3 + \pi/2 = \text{R.H.S. of equation}$

When $x = -1/2$

L.H.S. of equation = $\cos^{-1} (-3/2) + \cos^{-1} (-1/2) = \pi - \cos^{-1} (3/2) + \pi - \cos^{-1} (1/2)$

$= \pi - \pi/6 + \pi - \pi/3 = 3\pi/2 \neq \text{R.H.S. of equation}$

$\therefore x = 1/2$ is the only solution

Ex.2 Solve for x : $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$.

Sol.

$$\text{We have } (\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow (\tan^{-1} x + \cot^{-1} x)^2 - 2 \tan^{-1} x \cdot \cot^{-1} x = \frac{5\pi^2}{8}$$

$$\Rightarrow \frac{\pi^2}{4} - 2 \cdot \frac{\pi}{2} \cdot \tan^{-1} x + 2 (\tan^{-1} x)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow \left(\frac{\pi}{2}\right)^2 - 2 \tan^{-1} x \cdot (\pi/2 - \tan^{-1} x) = \frac{5\pi^2}{8}$$

$$\Rightarrow 2 (\tan^{-1} x)^2 - \pi \tan^{-1} x - \frac{3\pi^2}{8} = 0$$

$$\tan^{-1} x = -\pi/4, 3\pi/4 \Rightarrow \tan^{-1} x = -\pi/4; x = -$$

$$1, \text{ \{neglecting } \tan^{-1} x = 3\pi/4 \text{ as } \tan^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \}$$

Ex.3 Determine the integral values of 'k' for which the system, $(\arctan x)^2 + (\arccos y)^2 = \pi^2 k$ and $\tan^{-1} x + \cos^{-1} y = \pi/2$ posses solution and find all the solutions.

Sol.

$$\begin{aligned} 0 \leq (\tan^{-1} x)^2 \leq \frac{\pi^2}{4} \\ 0 \leq (\cos^{-1} y)^2 \leq \pi^2 \end{aligned} \Rightarrow$$

$$\Rightarrow (\tan^{-1} x)^2 + (\cos^{-1} y)^2 \leq \frac{5\pi^2}{4}$$

$$\text{hence } k\pi^2 \leq \frac{5\pi^2}{4} \Rightarrow k \leq \frac{5}{4} \quad \dots(1)$$

$$\text{Now put } \tan^{-1} x = \frac{\pi}{2} - \cos^{-1} y$$

$$\Rightarrow \left(\frac{\pi}{2} - \cos^{-1} y\right)^2 + (\cos^{-1} y)^2 = \pi^2 k$$

$$\Rightarrow 2t^2 - \pi t + \left(\frac{\pi^2}{4} - k\pi^2\right) = 0$$

$$\Rightarrow \pi^2 - 8 \left(\frac{\pi^2}{4} - k\pi^2\right) \geq 0 = 1 - 2 + 8k \geq 0 = k \geq 1/2 \dots(2)$$

From (1) and (2) $k = 1$

$$\therefore t = \frac{\pi \pm \sqrt{8\pi^2 - \pi^2}}{4} = \frac{\pi \pm \sqrt{7}\pi}{4} = (1 \pm \sqrt{7}) \frac{\pi}{4} \text{ or } \cos^{-1} y$$

$$= (\sqrt{7} + 1) \frac{\pi}{4} \text{ (as } 0 \leq \cos^{-1} y \leq \pi \text{)}$$

$$\therefore \tan^{-1} x = \frac{\pi}{2} - (\sqrt{7} + 1) \frac{\pi}{4}$$

$$= \frac{\pi}{4} [(1 - \sqrt{7})]$$

$$\Rightarrow x = \tan \left((1 - \sqrt{7}) \frac{\pi}{4} \right)$$

G. Inequations involving inverse trigonometric functions

Ex.1 Find the interval in which $\cos^{-1} x > \sin^{-1} x$.

Sol.

We have, $\cos^{-1} x > \sin^{-1} x$ {for $\cos^{-1} x$ to be real; $x \in [-1, 1]$ }

$$2 \cos^{-1} x > \pi/2 \Rightarrow \cos^{-1} x > \pi/4 \text{ or } \cos(\cos^{-1} x) < \cos \pi/4$$

$$\Rightarrow x < \frac{1}{\sqrt{2}}$$

$$\therefore x \in \left(-1, \frac{1}{\sqrt{2}} \right)$$

Ex.2 Find the solution set of the inequation $\sin^{-1}(\sin 5) > x^2 - 4x$

Sol.

$$\sin^{-1}(\sin 5) > x^2 - 4x \Rightarrow \sin^{-1}[\sin(5 - 2\pi)] > x^2 - 4x$$

$$\Rightarrow x^2 - 4x < 5 - 2\pi \Rightarrow x^2 - 4x + (2\pi - 5) < 0$$

$$\Rightarrow 2 - \sqrt{9 - 2\pi} < x < 2 + \sqrt{9 - 2\pi}$$

$$\Rightarrow x \in (2 - \sqrt{9 - 2\pi}, 2 + \sqrt{9 - 2\pi})$$

Summation of Series

Ex.1 Sum the series,

$$\tan^{-1} \frac{4}{1+3 \cdot 4} + \tan^{-1} \frac{6}{1+8 \cdot 9} + \tan^{-1} \frac{8}{1+15 \cdot 16} + \dots \text{ to 'n' terms.}$$

Sol.

$$T_n = \tan^{-1} \frac{2(n+1)}{1+\{(n+1)^2-1\}\{(n+1)^2\}}$$

$$= \tan^{-1} \frac{2n+2}{1+(n^2+2n)(n+1)^2}$$

$$= \tan^{-1} \left[\frac{2n+2}{1+n(n+2)(n+1)(n+1)} \right]$$

$$= \tan^{-1} \left[\frac{(n+1)(n+2)-n(n+1)}{1+\{n(n+1)\}\{(n+1)(n+2)\}} \right]$$

$$= \tan^{-1} (n+1)(n+2) - \tan^{-1} n(n+1)$$

Put $n = 1, 2, 3, \dots, n$ and add, we get $S_n = \tan^{-1} (n+1)(n+2) - \tan^{-1} 2$

Ex.2 Sum the series to 'n' terms, $\tan^{-1} \frac{2}{4} + \tan^{-1} \frac{2}{9} + \tan^{-1} \frac{2}{16} + \tan^{-1} \frac{2}{25} + \dots$
to 'n' terms. Also show that, $S_\infty = \tan^{-1} 3$.

Sol.

$$T_n = \tan^{-1} \frac{2}{n^2+2n+1}$$

$$= \tan^{-1} \frac{(n+2)-n}{1+n(n+2)}$$

$$= \tan^{-1} (n+2) - \tan^{-1} (n)$$

Hence, $S_n = \tan^{-1} (n+2) + \tan^{-1} (n+1) - (\tan^{-1} 1 + \tan^{-1} 2)$



$$S_{\infty} = \lim_{n \rightarrow \infty} S_n = \frac{\pi}{2} + \frac{\pi}{2} - \left(\pi + \tan^{-1} \left(\frac{1+2}{1-2} \right) \right) = \tan^{-1} 3$$

Ex.3 If the sum $\sum_{n=1}^{10} \sum_{m=1}^{10} \tan^{-1} \left(\frac{m}{n} \right) = k\pi$, find the value of k.

Sol.

$$S = \sum_{n=1}^{10} \left(\tan^{-1} \frac{1}{n} + \tan^{-1} \frac{2}{n} + \tan^{-1} \frac{3}{n} + \dots + \tan^{-1} \frac{10}{n} \right) \text{ now consider}$$

$$\sum_{n=1}^{10} \tan^{-1} \frac{1}{n} = \underbrace{\tan^{-1} 1}_{\tan^{-1} 1} + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} + \dots + \tan^{-1} \frac{1}{9} + \tan^{-1} \frac{1}{10}$$

$$\sum_{n=1}^{10} \tan^{-1} \frac{2}{n} = \tan^{-1} \frac{2}{1} + \underbrace{\tan^{-1} \frac{2}{2}}_{\tan^{-1} 1} + \tan^{-1} \frac{2}{3} + \tan^{-1} \frac{2}{4} + \tan^{-1} \frac{2}{5} + \dots + \tan^{-1} \frac{2}{10} \dots\dots$$

$$\sum_{n=1}^{10} \tan^{-1} \frac{10}{n} = \tan^{-1} \frac{10}{1} + \tan^{-1} \frac{10}{2} + \tan^{-1} \frac{10}{3} + \tan^{-1} \frac{10}{4} + \dots + \underbrace{\tan^{-1} \frac{10}{10}}_{\tan^{-1} 1}$$

$$S = \underbrace{\left(10 \cdot \frac{\pi}{4} \right)}_{\dots\dots\dots} + \underbrace{\left(\tan^{-1} \frac{1}{2} + \tan^{-1} 2 \right)}_{\dots\dots\dots} + \underbrace{\left(\tan^{-1} \frac{1}{3} + \tan^{-1} 3 \right)}_{\dots\dots\dots} + \dots\dots$$

$$S = \frac{5\pi}{2} + \frac{45\pi}{2} = \frac{50\pi}{2} = 25\pi \quad \Rightarrow \quad k = 25$$

Formulas: Inverse Trigonometric Functions

GENERAL DEFINITION(S): 1. $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ etc. denote angles or real numbers whose sine is x , whose cosine is x and whose tangent is x , provided that the answers given are numerically smallest available. These are also written as $\arcsin x$, $\arccos x$ etc.

If there are two angles one positive & the other negative having same numerical value, then positive angle should be taken.

2. PRINCIPAL VALUES AND DOMAINS OF INVERSE CIRCULAR FUNCTIONS :



(i) $y = \sin^{-1} x$ where $-1 \leq x \leq 1$; $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $\sin y = x$.

(ii) $y = \cos^{-1} x$ where $-1 \leq x \leq 1$; $0 \leq y \leq \pi$ and $\cos y = x$.

(iii) $y = \tan^{-1} x$ where $x \in \mathbb{R}$; $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\tan y = x$.

(iv) $y = \operatorname{cosec}^{-1} x$ where $x \leq -1$ or $x \geq 1$; $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$ and $\operatorname{cosec} y = x$

(v) $y = \sec^{-1} x$ where $x \leq -1$ or $x \geq 1$; $0 \leq y \leq \pi$; $y \neq \pi/2$ and $\sec y = x$.

(vi) $y = \cot^{-1} x$ where $x \in \mathbb{R}$, $0 < y < \pi$ and $\cot y = x$.

NOTE THAT : (a) 1st quadrant is common to all the inverse functions.

(b) 3rd quadrant is not used in inverse functions.

(c) 4th quadrant is used in the CLOCKWISE DIRECTION i.e. $-\pi/2 \leq y \leq 0$.

3. PROPERTIES OF INVERSE CIRCULAR FUNCTIONS:

P-1 (i) $\sin (\sin^{-1} x) = x$, $-1 \leq x \leq 1$

(ii) $\cos (\cos^{-1} x) = x$, $-1 \leq x \leq 1$

(iii) $\tan (\tan^{-1} x) = x$, $x \in \mathbb{R}$

(iv) $\sin^{-1} (\sin x) = x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

(v) $\cos^{-1} (\cos x) = x$; $0 \leq x \leq \pi$

(vi) $\tan^{-1} (\tan x) = x$; $-\frac{\pi}{2} < x < \frac{\pi}{2}$

P-2 (i) $\operatorname{cosec}^{-1} x = \sin^{-1}(1/x)$; $x \leq -1$, $x \geq 1$

(ii) $\sec^{-1} x = \cos^{-1}(1/x)$; $x \leq -1$, $x \geq 1$

(iii) $\cot^{-1} x = \tan^{-1} (1/x)$; $x > 0 = \pi + \tan^{-1}(1/x)$; $x < 0$



P-3 (i) $\sin^{-1}(-x) = -\sin^{-1}x, -1 \leq x \leq 1$

(ii) $\tan^{-1}(-x) = -\tan^{-1}x, x \in \mathbb{R}$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1}x, -1 \leq x \leq 1$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in \mathbb{R}$

P-4 (i) $\sin^{-1}x + \cos^{-1}x = \pi/2, -1 \leq x \leq 1$

(ii) $\tan^{-1}x + \cot^{-1}x = \pi/2, x \in \mathbb{R}$

(iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \pi/2, |x| \geq 1$

P-5 $\tan^{-1}x + \tan^{-1}y = \tan^{-1} \frac{x+y}{1-xy}$ where $x > 0, y > 0$ & $xy < 1$

$= \pi + \tan^{-1} \frac{x+y}{1-xy}$ where $x > 0, y > 0$ & $xy > 1$

$\tan^{-1}x - \tan^{-1}y = \tan^{-1} \frac{x-y}{1+xy}$ where $x > 0, y > 0$

P-6 (i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1} \left[x\sqrt{1-y^2} + y\sqrt{1-x^2} \right]$ where $x \geq 0, y \geq 0$ & $(x^2 + y^2) \leq 1$

Note that : $x^2 + y^2 \leq 1 \Rightarrow 0 \leq \sin^{-1}x + \sin^{-1}y \leq \pi/2$

(ii) $\sin^{-1}x + \sin^{-1}y = \pi - \sin^{-1} \left[x\sqrt{1-y^2} + y\sqrt{1-x^2} \right]$ where $x \geq 0, y \geq 0$ & $x^2 + y^2 > 1$

Note that : $x^2 + y^2 > 1 \Rightarrow \pi/2 < \sin^{-1}x + \sin^{-1}y < \pi$

(iii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1} \left[x\sqrt{1-y^2} - y\sqrt{1-x^2} \right]$ where $x > 0, y > 0$

(iv) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1} \left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2} \right]$ where $x \geq 0, y \geq 0$

P-7 If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1} \left[\frac{x+y+z-xyz}{1-xy-yz-zx} \right]$ if, $x > 0, y > 0, z > 0$ & $xy + yz + zx < 1$

Note : (i) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$ then $x + y + z = xyz$

(ii) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi/2$ then $xy + yz + zx = 1$

P-8 $2 \tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2} = \cos^{-1} \frac{1-x^2}{1+x^2} = \tan^{-1} \frac{2x}{1-x^2}$ Note very carefully that :

$$\sin^{-1} \frac{2x}{1+x^2} = \begin{cases} 2 \tan^{-1} x & \text{if } |x| \leq 1 \\ \pi - 2 \tan^{-1} x & \text{if } x > 1 \\ -(\pi + 2 \tan^{-1} x) & \text{if } x < -1 \end{cases} \quad \cos^{-1} \frac{1-x^2}{1+x^2} = \begin{cases} 2 \tan^{-1} x & \text{if } x \geq 0 \\ -2 \tan^{-1} x & \text{if } x < 0 \end{cases}$$

$$\tan^{-1} \frac{2x}{1-x^2} = \begin{cases} 2 \tan^{-1} x & \text{if } |x| < 1 \\ \pi + 2 \tan^{-1} x & \text{if } x < -1 \\ -(\pi - 2 \tan^{-1} x) & \text{if } x > 1 \end{cases}$$

REMEMBER THAT : (i) $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = 3\pi/2 \Rightarrow x = y = z = 1$

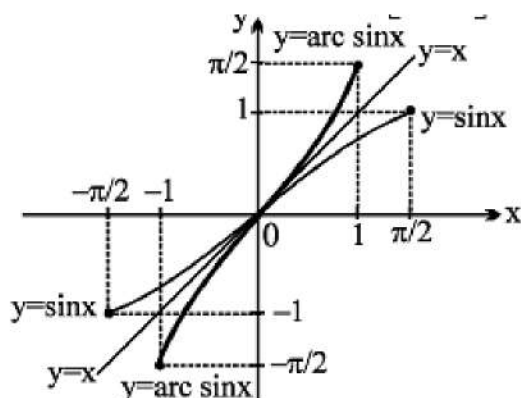
(ii) $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi \Rightarrow x = y = z = -1$

(iii) $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$ and $\tan^{-1} 1 + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{2}$

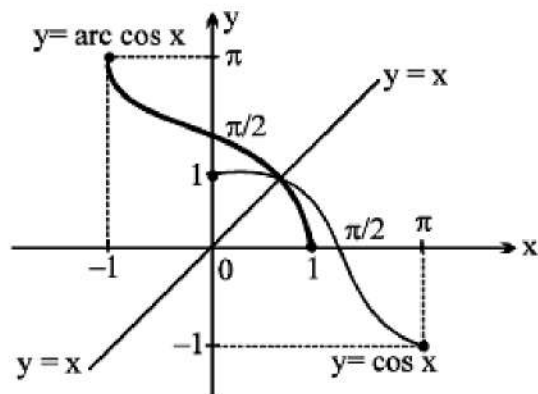
INVERSE TRIGONOMETRIC FUNCTIONS

SOME USEFUL GRAPHS

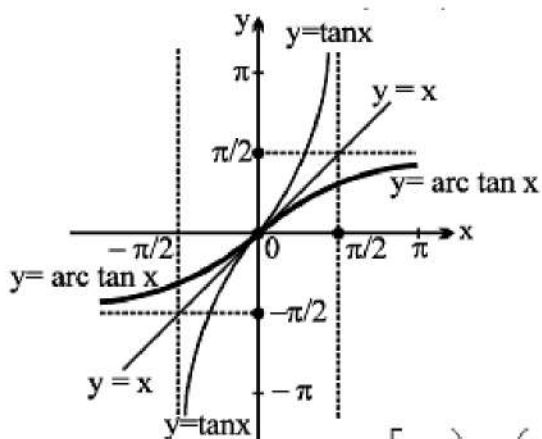
1. $y = \sin^{-1} x, |x| \leq 1, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



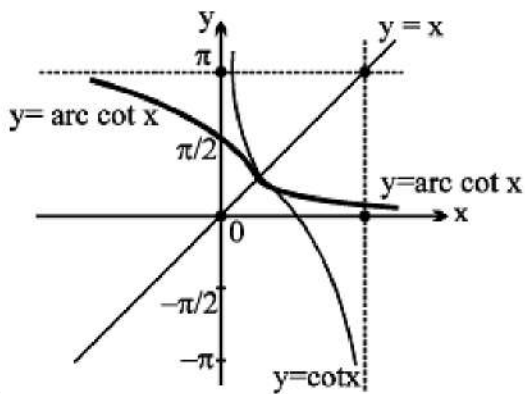
2. $y = \cos^{-1} x, |x| \leq 1, y \in [0, \pi]$



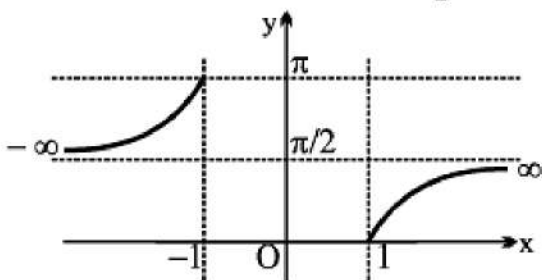
3. $y = \tan^{-1} x, x \in \mathbb{R}, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



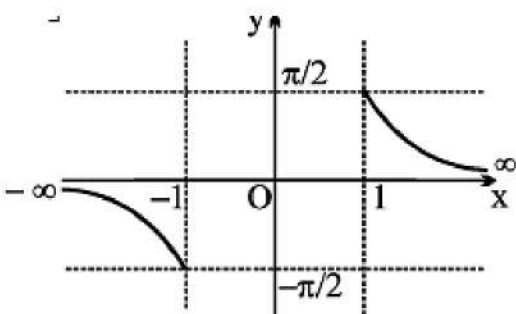
4. $y = \cot^{-1} x, x \in \mathbb{R}, y \in (0, \pi)$



5. $y = \sec^{-1} x, |x| \geq 1, y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

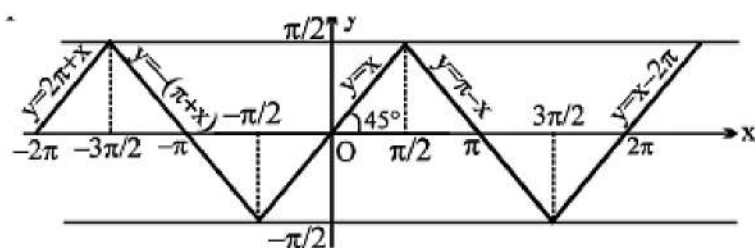


6. $y = \operatorname{cosec}^{-1} x, |x| \geq 1, y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

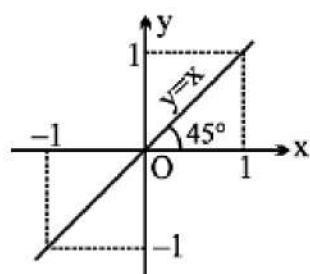


7. (a) $y = \sin^{-1}(\sin x), x \in \mathbb{R}, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

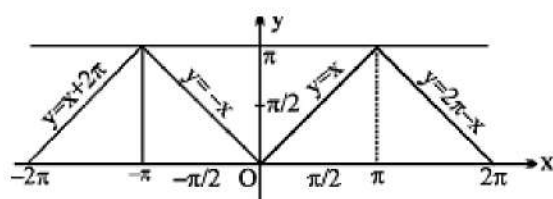
Periodic with period 2π



(b) $y = \sin(\sin^{-1} x)$, $x \in [-1, 1]$, $y \in [-1, 1]$, y is aperiodic

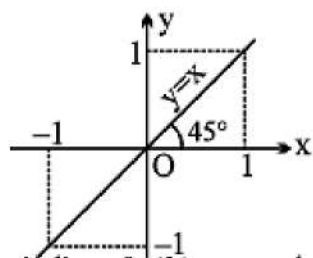


8. (a) $y = \cos^{-1}(\cos x)$, $x \in \mathbb{R}$, $y \in [0, \pi]$, $y = x$ periodic with period 2π



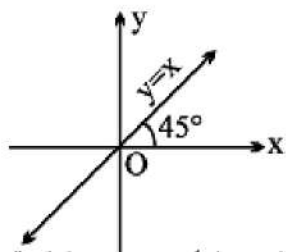
(b) $y = \cos(\cos^{-1} x)$, $y = x$

$x \in [-1, 1]$, $y \in [-1, 1]$, y is aperiodic

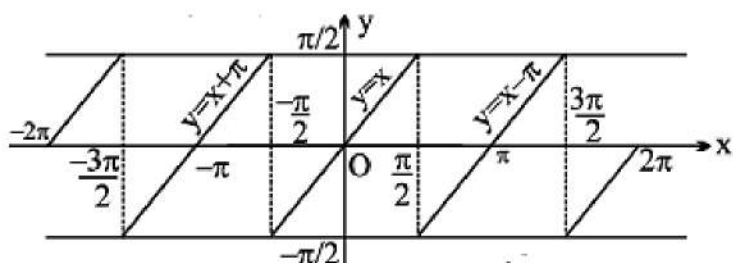


9. (a) $y = \tan(\tan^{-1} x)$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, y is aperiodic $= x$

$x \in \mathbb{R} - \left\{ (2n-1)\frac{\pi}{2} \mid n \in \mathbb{I} \right\}$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, periodic with period π

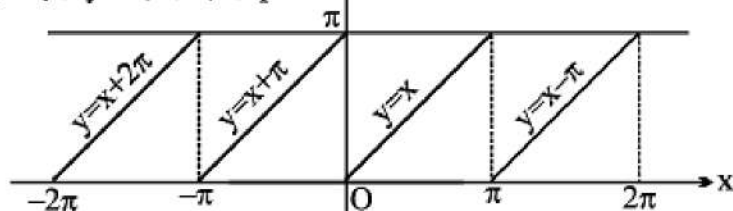


(b) $y = \tan^{-1}(\tan x) = x$



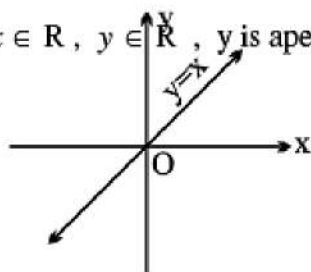
10. (a) $y = \cot^{-1}(\cot x) = x$

$x \in \mathbb{R} - \{n\pi\}$, $y \in (0, \pi)$, periodic with π



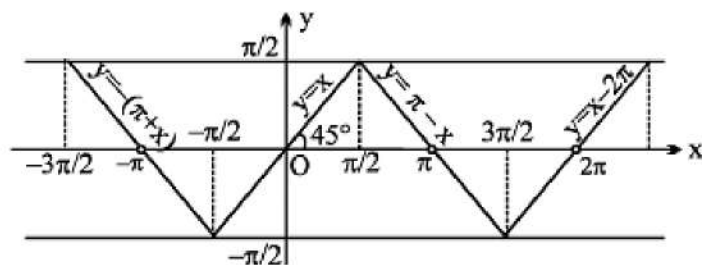
(b) $y = \cot(\cot^{-1} x) = x$

$x \in \mathbb{R}$, $y \in \mathbb{R}$, y is aperiodic



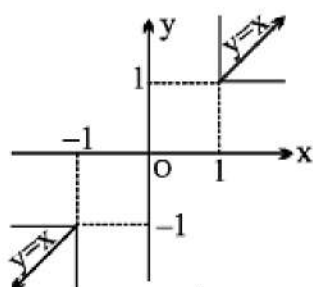
11. (a) $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x) = x$

$x \in \mathbb{R} - \{n\pi, n \in \mathbb{I}\}, y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
 y is periodic with period 2π



(b) $y = \operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$

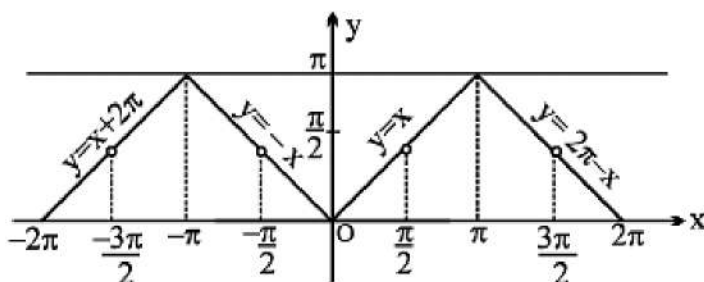
$|x| \geq 1, |y| \geq 1, y$ is aperiodic



12. (a) $y = \sec^{-1}(\sec x) = x$

y is periodic with period 2π ;

$x \in \mathbb{R} - \{(2n-1)\frac{\pi}{2}, n \in \mathbb{I}\}, y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



(b) $y = \sec(\sec^{-1} x) = x$

$|x| \geq 1, |y| \geq 1, y$ is aperiodic

